SCALAR DIFFERENTIAL INVARIANTS OF MONGE–AMPÈRE EQUATIONS

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Abstract. In this paper, we construct scalar differential invariants of Monge–Ampère equations in general position.

1. Introduction

With this paper we start a systematic study of differential invariants of Monge-Ampère equations aiming at the classification problem, methods of integrations and other applications. We are interested in equations in two independent variables (the classical case). Monge-Ampère equations merit a special attention due to a large spectrum of various applications, first of all, in differential geometry and mathematical physics. Moreover, they form a natural testing area for new methods emerging in the modern theory of nonlinear PDE's.

In spite of more than 200 years of history of Monge-Ampère equations and numerous publications devoted to them it would be an exaggeration to say that their nature is well understood. An important success was establishing of the existence and uniqueness theorems for them (see [6, 3] for local aspects and [10] for global ones). Modern formulation of the classical Monge's method of solution was given by Matsuda [7, 8] and Morimoto [9]. The hopes are that differential invariants could illuminate these and many other aspects of the theory of Monge-Ampère equations.

According to [12] (see also [1]) scalar differential invariants provide key to solving the classification problem for any kind of geometrical structures. In fact, geometrical structures of a given type are classified by solutions of a naturally associated *classifying* (differential) equation, which describes interrelations among the corresponding scalar differential invariants. More exactly, scalar differential invariant are smooth functions on the *classifying diffiety*, which is the infinite prolongation of the classifying equation. This diffiety has, generally, singularities and its singular strata classify those geometrical structures that possess nontrivial symmetries. Each of these strata is also an infinitely prolonged differential equation in a lesser number of independent variables. For instance, homogeneous structures correspond to the zero-dimensional case. So, the classification problem consists of a complete description of all strata composing the classifying diffiety and, therefore,

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presuppose a complete symmetry analysis of the geometric structures under consideration. The interested reader will find an illustration of the above said in [13] where plane 3-webs, a rather simple geometrical structure, is considered.

In this paper we interpret Monge-Ampère equations as certain geometrical structures on 5-dimensional contact manifolds and look for their differential invariants, not only scalar, with respect the group of contact transformations. Here we limit ourself to the case of generic hyperbolic equations. This is because of two reasons. First, the study of singular strata presupposes that of the generic one. Second, for the hyperbolic equations differential invariants are easier visible due to existence of bicharacteristics.

Differential invariants found in this paper give a solution of the classification problem for generic hyperbolic equations. Unfortunately, this solution is of a theoretical nature and requires a substantial computer support in analysis of concrete cases. So, a further simplification work is necessary to improve its efficiency.

Differential invariants for elliptic and parabolic Monge-Ampère equations can be obtained more or less straightforwardly by following the approach developed in this paper. This and a study of singular strata will be the subject of subsequent publications.

2. Preliminaries

Below, all manifolds and maps are supposed to be smooth. By $[f]_p^k$, $k = 0, 1, 2, \ldots, \infty$, we denote the k-jet of a map f at a point p, by $\mathbb{R}$ we denote the field of real numbers, and by $\mathbb{R}^n$ we denote the n-dimensional arithmetic space.

2.1. **Jet bundles.**

Here we recall necessary definitions and facts about jet bundles, see [4].

Let M be an n-dimensional manifold, let E be an $n+m$-dimensional manifold, and let

$$\pi : E \longrightarrow M .$$

be a bundle. By

$$\pi_k : J^k\pi \to M , \quad \pi_k : [S]_p^k \mapsto p , \quad k = 0, 1, 2, \ldots$$

denote the bundle of all k-jets of sections of π. For any $l > m \geq 0$, the natural projection is defined as

$$\pi_{l,m} : J^l\pi \to J^m\pi , \quad \pi_{l,m} : [S]_p^l \mapsto [S]_p^m .$$

Any section S of π generates the section j_kS of the bundle π_k by the formula

$$j_kS : p \mapsto [S]_p^k .$$

By definition, put

$$L_S^k = \operatorname{Im} j_kS .$$

Let θ_{k+1} be an arbitrary point of $J^{k+1}\pi$, $\theta_k = \pi_{k+1,k}(\theta_{k+1})$, and $T_{\theta_k}(J^k\pi)$ the tangent space to $J^k\pi$ at the point θ_k. Then θ_{k+1} defines the subspace $K_{\theta_{k+1}} \subset T_{\theta_k}(J^k\pi)$ by the formula

$$K_{\theta_{k+1}} = T_{\theta_k}(L_S^k) .$$

Clearly, the correspondence $\theta_{k+1} \mapsto K_{\theta_{k+1}}$ is a bijection. Therefore we identify θ_{k+1} with $K_{\theta_{k+1}}$. It is easy to prove that

$$T_{\theta_k}(J^k\pi) = K_{\theta_{k+1}} \oplus T_{\theta_k}(\pi_k^{-1}(p))\,. \tag{1}$$

Consider all submanifolds of the form L_S^k containing θ_k. Subspace spanned by their tangent spaces $T_{\theta_k}(L_S^k)$ is denoted by $\mathcal{C}(\theta_k)$ and it is called the *Cartan plane at θ_k*. The distribution

$$\mathcal{C}_k : \theta_k \mapsto \mathcal{C}(\theta_k)$$

is called the *Cartan distribution on $J^k\pi$*. The distribution $\mathcal{C}_k$, $k \geq 1$, can be defined as the kernel of the *Cartan form*

$$U_k = \mathrm{pr}_2 \circ (\pi_{k,k-1})_*\,,$$

where $\mathrm{pr}_2 : T_{\theta_{k-1}}(J^{k-1}\pi) \to T_{\theta_{k-1}}(\pi_{k-1}^{-1}(p))$ is the projection generated by direct sum decomposition (1).

2.2. **The contact structure.** Consider the trivial bundle

$$\tau : \mathbb{R}^2 \times \mathbb{R} \longrightarrow \mathbb{R}^2\,, \quad \tau : (\,x, y, z\,) \mapsto (\,x, y\,)\,.$$

By x, y, z, $p = z_x$, $q = z_y$, $r = z_{xx}$, $s = z_{xy}$, $t = z_{yy}$ we denote the standard coordinates in $J^2\tau$.

The Cartan distribution C_1 on $J^1\tau$ is a *contact structure* on $J^1\tau$. The corresponding contact 1-form U_1 has the canonical form

$$U_1 = dz - p\,dx - q\,dy\,.$$

in the standard coordinates.

A diffeomorphism $\varphi : J^1\tau \to J^1\tau$ is called a *contact transformation* if it preserves the Cartan distribution. Obviously, a diffeomorphism φ is a contact transformation iff there exist a nowhere vanishing function λ such that

$$\varphi^*(U_1) = \lambda\, U_1\,.$$

Any contact transformation φ can be lifted to the diffeomorphism

$$\varphi_\tau^{(1)} : J^2\tau \longrightarrow J^2\tau$$

by the formula

$$\varphi_\tau^{(1)} : \theta_2 \equiv K_{\theta_2} \mapsto \varphi_*(K_{\theta_2}) \equiv \tilde{\theta}_2 = \varphi_\tau^{(1)}(\theta_2)\,.$$

If φ is defined on an open set $V \subset J^1\tau$, then $\varphi_\tau^{(1)}$ is defined on an open, everywhere dense subset of $\tau_{2,1}^{-1}(V)$.

A vector field X in $J^1\tau$ is a *contact vector field* if its flow φ_t consists of contact transformations. Clearly, X is a contact vector field iff there exist a function λ such that

$$L_X(U_1) = \lambda\, U_1\,,$$

where L_X is the Lie derivative with respect to X.

There exists a natural bijection of the set of all contact vector fields in $J^1\tau$ onto the set of all functions in $J^1\tau$. It is defined by the formula

$$X \mapsto f = X \lrcorner\, U_1\,.$$

The function $f = X \rfloor U_1$ is called the *generating function of a contact vector field* X. The contact vector field X corresponding to f is denoted by X_f. In the standard coordinates, the field X_f is given by the formula

$$X_f = -f_p \frac{\partial}{\partial x} - f_q \frac{\partial}{\partial y} + (f - pf_p - qf_q) \frac{\partial}{\partial z} + (f_x + pf_z) \frac{\partial}{\partial p} + (f_y + qf_z) \frac{\partial}{\partial q} \quad (2)$$

2.3. **Operations over vector-valued forms.** Let M be a smooth n-dimensional manifold, $\Lambda^i(M)$ the $C^\infty(M)$-module of differential i-forms on M, $i = 1, 2, \ldots$, and $D(M)$ the $C^\infty(M)$-module of vector fields on M. Let $\alpha \in \Lambda^k(M)$, $\beta \in \Lambda^r(M)$, and $X, Y \in D(M)$. Then the Frölicher–Nijenhuis bracket $[\![\cdot\,,\cdot]\!]$ of the vector-valued forms $\alpha \otimes X$ and $\beta \otimes Y$ is defined by the formula

$$\begin{aligned} &[\![\alpha \otimes X,\, \beta \otimes Y]\!] \\ &\qquad = \alpha \wedge \beta \otimes \bigl[X, Y\bigr] + \alpha \wedge X(\beta) \otimes Y - Y(\alpha) \wedge \beta \otimes X \\ &\qquad\qquad + (-1)^k d\alpha \wedge (X \rfloor \beta) \otimes Y - (-1)^k (Y \rfloor \alpha) \wedge d\beta \otimes X\,, \end{aligned}$$

see [2]. The contraction $\rfloor$ of forms $\alpha \otimes X$ and $\beta \otimes Y$ is defined by the formula

$$(\alpha \otimes X) \rfloor (\beta \otimes Y) = \alpha \wedge (X \rfloor \beta) \otimes Y\,.$$

2.4. **Distributions and their curvatures.** The following simple construction allows one to associate a vector valued 2-form with a projector. Namely, let $P, Q \in D(M)$ be endomorphisms of the $C^\infty(M)$-module $D(M)$ such that $QP = 0$. Then

$$\Omega_{Q,P}(X, Y) = Q[P(X), P(Y)], \quad X, Y \in D(M), \tag{3}$$

obviously, is skew-symmetric and $C^\infty(M)$-bilinear, i.e., a vector valued form. More precisely, it takes values in Im $Q \subset D(M)$. If $P : D(M) \to D(M)$ is a projector, i.e., $P^2 = P$, then the associated *curvature form* of P is defined to be

$$\mathcal{R}_P = \Omega_{I-P,P} \tag{4}$$

with $I = \mathrm{id}_{D(M)}$.

Let $\mathcal{D}$ be a distribution on M. Then by $\mathcal{D}^{(1)}$ we denote the distribution generated by all vector fields X and $[X, Y]$, where $X, Y \in \mathcal{D}$. Setting $\mathcal{D}^{(0)} = \mathcal{D}$, we define $\mathcal{D}^{(r+1)}$, $r = 0, 1, \ldots$, inductively by the formula $\mathcal{D}^{(r+1)} = (\mathcal{D}^{(r)})^{(1)}$.

3. Hyperbolic Monge–Ampère equations

3.1. **Monge–Ampère equations.** The Monge–Ampère equation is a partial differential equation of the form

$$N(z_{xx}z_{yy} - z_{xy}^2) + Az_{xx} + Bz_{xy} + Cz_{yy} + D = 0\,, \tag{5}$$

where x, y are independent variables, z is a dependent variable, $z_{xx} = \partial^2 z/\partial x^2$, $z_{xy} = \partial^2 z/\partial x\,\partial y$, $z_{yy} = \partial^2 z/\partial y^2$, and coefficients N, A, B, C, D are functions of x, y, z, $z_x = \partial z/\partial x$ and $z_y = \partial z/\partial y$.

We identify equation (5) with the submanifold $\mathcal{E}$ of the jet bundle $J^2\tau$ determined by the equation

$$N(rt - s^2) + Ar + Bs + Ct + D = 0\,. \tag{6}$$

Obviously,

$$\tau_{2,1}(\mathcal{E}) = J^1\tau\,.$$

Let $\theta_2 \in \mathcal{E}$, $\tau_{2,1}(\theta_2) = \theta_1$, and F_{θ_1} be the fibre of the projection $\tau_{2,1}$ over the point $\theta_1 \in J^1\tau$. Then the subspace

$$\operatorname{Smbl}_{\theta_2}\mathcal{E} = T_{\theta_2}\mathcal{E} \cap T_{\theta_2}F_{\theta_1},$$

where $T_{\theta_2}\mathcal{E}$ is the tangent space to $\mathcal{E}$ at θ_2 is called the *symbol of the equation* $\mathcal{E}$ *at the point* $\theta_2 \in \mathcal{E}$. In terms of standard coordinates, $\operatorname{Smbl}_{\theta_2}\mathcal{E}$ is described by the linear equation

$$N(t\tilde{r} + r\tilde{t} - 2s\tilde{s}) + A\tilde{r} + B\tilde{s} + C\tilde{t} = 0, \tag{7}$$

where $\tilde{r}$, $\tilde{s}$, $\tilde{t}$ are the standard coordinates in T_{θ_2} generated by the standard coordinates on $J^2\tau$.

A point $\theta_2 \in \mathcal{E}$ can be elliptic, parabolic, or hyperbolic. To introduce these notions, let us consider a one-dimensional subspace $P \subset \mathcal{C}(\theta_1)$ such that $(\tau_1)_*P \neq 0$. By definition, put

$$l(P) = \{\,\theta_2 \in F_{\theta_1} \bigm| P \subset K_{\theta_2}\,\}\,.$$

The submanifold $l(P)$ is called a *1-ray*. In terms of standard coordinates, let $\theta_1 = (x, y, z, p, q)$, $P = \langle v\rangle$ and

$$v = \zeta_1\frac{\partial}{\partial x} + \zeta_2\frac{\partial}{\partial y} + \mu\frac{\partial}{\partial z} + \eta_1\frac{\partial}{\partial p} + \eta_2\frac{\partial}{\partial q}\,. \tag{8}$$

Then $(\tau_1)_*P \neq 0$ means that

$$(\zeta_1, \zeta_2) \neq (0, 0)\,, \tag{9}$$

$v \in \mathcal{C}(\theta_1)$ means that

$$\mu = \zeta_1 p + \zeta_2 q\,, \tag{10}$$

and $P \subset K_{\theta_2}$ means that

$$\begin{cases} \eta_1 = \zeta_1 r + \zeta_2 s\,, \\ \eta_2 = \zeta_1 s + \zeta_2 t\,, \end{cases} \tag{11}$$

where r, s, t are the standard coordinates of θ_2 in the fibre F_{θ_1}. From system (11), we see that $l(P)$ is an affine straight line in F_{θ_1}. By $\ell_{\theta_2}(P)$ we denote the tangent space $T_{\theta_2}l(P)$ to $l(P)$ at the point $\theta_2 \in l(P)$. We call it a *1-ray subspace*. In terms of the standard coordinates $\tilde{r}, \tilde{s}, \tilde{t}$ in $T_{\theta_2}F_{\theta_1}$, vectors of $\ell_{\theta_2}(P)$ satisfy

$$\begin{cases} \zeta_1\tilde{r} + \zeta_2\tilde{s} = 0\,, \\ \zeta_1\tilde{s} + \zeta_2\tilde{t} = 0\,, \end{cases} \tag{12}$$

Obviously, $\ell_{\theta_2}(P)$ is spanned by the vector

$$(\,\tilde{r},\, \tilde{s},\, \tilde{t}\,) = (\,\zeta_2^2,\, -\zeta_1\zeta_2,\, \zeta_1^2\,)\,. \tag{13}$$

Taking into account (9), we observe that all 1-ray subspaces form the cone

$$\mathcal{V}_{\theta_2} = \{\,\tilde{r}\tilde{t} - \tilde{s}^2 = 0\,\}$$

in the tangent space $T_{\theta_2}F_{\theta_1}$. This cone is called the *cone of singular square forms*. Obviously, the intersection $\operatorname{Smbl}_{\theta_2}\mathcal{E} \cap \mathcal{V}_{\theta_2}$ is either zero or a single 1-ray subspace or two 1-ray subspaces. Correspondingly, the point $\theta_2 \in \mathcal{E}$ is then called *elliptic* or *parabolic* or *hyperbolic*. It is not hard to prove that a contact transformation takes an elliptic, parabolic, or hyperbolic point to an elliptic, parabolic, or hyperbolic point, respectively. The equation $\mathcal{E}$ is called *elliptic*, *parabolic* or *hyperbolic* if all its points are elliptic, parabolic or hyperbolic, respectively. In this work, we consider hyperbolic Monge–Ampère equations only. It is easy to prove that $\mathcal{E}$ is hyperbolic iff its coefficients satisfy the condition

$$\Delta = B^2 - 4AC + 4ND > 0\,. \tag{14}$$

3.2. **Skew-orthogonal distributions.** Following [11], we show that a hyperbolic Monge–Ampère equation is equivalent to a pair of skew-orthogonal two-dimensional distributions in the Cartan distribution on $J^1\tau$.

Let θ_1 be an arbitrary point of $J^1\tau$. By $\mathcal{Q}_{\theta_1}$ we denote the union of all one-dimensional subspaces P of $\mathcal{C}(\theta_1)$ such that $\tau_* P \neq 0$ and the 1-ray $l(P)$ is tangent to $\mathcal{E}$ at least at one point.

Proposition 3.1. *Let $\mathcal{E}$ be a hyperbolic Monge–Ampère equation. Then $\mathcal{Q}_{\theta_1}$ is the union of two-dimensional subspaces $\mathcal{D}^1_{\mathcal{E}}(\theta_1)$ and $\mathcal{D}^2_{\mathcal{E}}(\theta_1)$ of the Cartan plane $\mathcal{C}(\theta_1)$, so that*

1. $\mathcal{C}(\theta_1) = \mathcal{D}^1_{\mathcal{E}}(\theta_1) \oplus \mathcal{D}^2_{\mathcal{E}}(\theta_1)$,
2. $\mathcal{D}^1_{\mathcal{E}}(\theta_1)$ *and* $\mathcal{D}^2_{\mathcal{E}}(\theta_1)$ *are skew-orthogonal with respect to the symplectic form* $dU_1 = dp \wedge dx + dq \wedge dy$ *on* $\mathcal{C}$.

Proof. We prove this proposition for Monge–Ampère equations such that $N \neq 0$. The proof for $N = 0$ follows from the fact that every Monge–Ampère equation can be transformed to one with $N \neq 0$ by an appropriate contact transformation.

Let $v \in \mathcal{Q}_{\theta_1}$ and $P = \langle v \rangle$. The condition for $l(P)$ to be tangent to $\mathcal{E}$ can be written in the following way. Suppose v is described by equation (8). Then the vector $(\zeta_2^2, -\zeta_1\zeta_2, \zeta_1^2)$ is tangent to $l(P)$. Now using (7) we deduce that $l(P)$ is tangent to $\mathcal{E}$ iff

$$M(r\zeta_1^2 + 2s\zeta_1\zeta_2 + t\zeta_2^2) + A\zeta_2^2 - B\zeta_1\zeta_2 + C\zeta_1^2 = 0\,.$$

Taking into account that the coordinates ζ_i and η_i of v are connected by equations (11), we reduce this equation to the form

$$M(\zeta_1\eta_1 + \zeta_2\eta_2) + A\zeta_2^2 - B\zeta_1\zeta_2 + C\zeta_1^2 = 0\,. \tag{15}$$

Taking into account that ζ_1 and ζ_2 are connected by equation (9), we put $\zeta_1 \neq 0$ (the case $\zeta_2 \neq 0$ is analogous). Then from (11) we get

$$r = \frac{1}{\zeta_1^2}(\eta_1\zeta_1 - \eta_2\zeta_2 + \zeta_2^2 t)\,, \quad s = \frac{1}{\zeta_1}(\eta_2 - \zeta_2 t)\,.$$

Substituting these expressions for r and s in equation (6) and taking into account equation (15), we obtain the equation

$$M\eta_2^2 + (A\zeta_2 - B\zeta_1)\eta_2 - A\zeta_1\eta_1 - D\zeta_1^2 = 0. \tag{16}$$

Solving the system of equations (15) and (16) with respect to η_1 and η_2, we obtain that

$$\eta_1 = \frac{(B \mp \sqrt{\Delta})\zeta_2 - 2C\zeta_1}{2M}\,, \quad \eta_2 = \frac{(B \pm \sqrt{\Delta})\zeta_1 - 2A\zeta_2}{2M}\,.$$

Hence, taking into account equation (10),

$$v = \zeta_1\left(\frac{\partial}{\partial x} + p\frac{\partial}{\partial z} - \frac{C}{N}\frac{\partial}{\partial p} + \frac{B \pm \sqrt{\Delta}}{2N}\frac{\partial}{\partial q}\right) + \zeta_2\left(\frac{\partial}{\partial y} + q\frac{\partial}{\partial z} + \frac{B \mp \sqrt{\Delta}}{2N}\frac{\partial}{\partial p} - \frac{A}{N}\frac{\partial}{\partial q}\right). \quad (17)$$

It follows now that $\mathcal{Q}_{\theta_1} = \langle X_1, X_2\rangle \cup \langle X_3, X_4\rangle$, where

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x} + p\frac{\partial}{\partial z} - \frac{C}{N}\frac{\partial}{\partial p} + \frac{B - \sqrt{\Delta}}{2N}\frac{\partial}{\partial q}, \\ X_2 &= \frac{\partial}{\partial y} + q\frac{\partial}{\partial z} + \frac{B + \sqrt{\Delta}}{2N}\frac{\partial}{\partial p} - \frac{A}{N}\frac{\partial}{\partial q}, \\ X_3 &= \frac{\partial}{\partial x} + p\frac{\partial}{\partial z} - \frac{C}{N}\frac{\partial}{\partial p} + \frac{B + \sqrt{\Delta}}{2N}\frac{\partial}{\partial q}, \\ X_4 &= \frac{\partial}{\partial y} + q\frac{\partial}{\partial z} + \frac{B - \sqrt{\Delta}}{2N}\frac{\partial}{\partial p} - \frac{A}{N}\frac{\partial}{\partial q}. \end{aligned} \quad (18)$$

Put

$$\mathcal{D}^1_{\mathcal{E}}(\theta_1) = \langle X_1, X_2\rangle, \quad \mathcal{D}^2_{\mathcal{E}}(\theta_1) = \langle X_3, X_4\rangle.$$

It is easy to check that $\mathcal{D}^1_{\mathcal{E}}(\theta_1)$ and $\mathcal{D}^2_{\mathcal{E}}(\theta_1)$ are skew-orthogonal and $\mathcal{D}^1_{\mathcal{E}}(\theta_1) \cap \mathcal{D}^2_{\mathcal{E}}(\theta_1) = \{0\}$. This completes the proof. □

From (18) we see that for a Monge–Ampère equation such that $N \neq 0$, the map τ_{1*} projects $\mathcal{D}^1_{\mathcal{E}}(\theta_1)$ and $\mathcal{D}^2_{\mathcal{E}}(\theta_1)$ onto the tangent space to the base of the bundle τ without degeneration.

It should be noted that if $N = 0$ (that is, if $\mathcal{E}$ is a quasilinear second order PDE), then the projections $\tau_{1*}\big(\mathcal{D}^1_{\mathcal{E}}(\theta_1)\big)$ and $\tau_{1*}\big(\mathcal{D}^2_{\mathcal{E}}(\theta_1)\big)$ are one-dimensional.

Thus an arbitrary hyperbolic Monge–Ampère equation generates two 2-dimensional skew-orthogonal subdistributions of the Cartan distribution $\mathcal{C}_1$ in $J^1\tau$.

Proposition 3.2. *Let $\mathcal{E}$ be a hyperbolic Monge–Ampère equation. Then $\theta_2 \in \mathcal{E}$ if and only if one of the following equivalent conditions holds:*

(1) $K_{\theta_2} \cap \mathcal{D}^1(\mathcal{E})_{\theta_1}$ *is a straight line,*
(2) $K_{\theta_2} \cap \mathcal{D}^2(\mathcal{E})_{\theta_1}$ *is a straight line.*

Proof. As in the proof of Proposition 3.1, it is enough to prove this proposition for the case $N \neq 0$.

Let $\theta_2 \in \mathcal{E}$. Then $\operatorname{Smbl}_{\theta_2} \mathcal{E} \cap \mathcal{V}_{\theta_2} = \ell_{\theta_2}(\langle v\rangle) \cup \ell_{\theta_2}(\langle \tilde v\rangle)$, where $\ell_{\theta_2}(\langle v\rangle)$ and $\ell_{\theta_2}(\langle \tilde v\rangle)$ are different straight lines. It follows that v and $\tilde v$ are different vectors of K_{θ_2}. They are skew-orthogonal because K_{θ_2} is a Lagrangian plane in $\mathcal{C}(\theta_1)$. From definition of $\mathcal{Q}_{\theta_1}$ we get $v, \tilde v \in \mathcal{Q}_{\theta_1}$. This means that K_{θ_2} intersects one of the planes $\mathcal{D}^1(\mathcal{E})_{\theta_1}$ and $\mathcal{D}^1(\mathcal{E})_{\theta_1}$ along $\langle v\rangle$ and the other one along $\langle \tilde v\rangle$.

Let θ_2 be a point of $J^2\tau$ such that K_{θ_2} intersects the plane $\mathcal{D}^1(\mathcal{E})_{\theta_1}$ along a straight line, that is, $K_{\theta_2} \cap \mathcal{D}^1(\mathcal{E})_{\theta_1} = \langle v\rangle$. Substituting coordinates η_1, η_2

of the vector v given by formula (17) into eq. (11), we obtain

$$\begin{aligned}\left(r+\frac{C}{N}\right)\zeta_1+\left(s-\frac{B-\sqrt{\Delta}}{2N}\right)\zeta_2 &= 0,\\ \left(s-\frac{B+\sqrt{\Delta}}{2N}\right)\zeta_1+\left(r+\frac{A}{N}\right)\zeta_2 &= 0.\end{aligned}$$

This system is necessarily singular (cf. (9)), whence its determinant is zero, which is exactly equation (6). Thus, $\theta_2 \in \mathcal{E}$. The case of $\mathcal{D}^2(\mathcal{E})_{\theta_1}$ differs only by the sign at $\sqrt{\Delta}$. □

It follows from this proposition that a hyperbolic Monge–Ampère equation $\mathcal{E}$ can be completely reconstructed from any of the associated distributions $\mathcal{D}^i(\mathcal{E})$, $i = 1, 2$.

Thus, every hyperbolic Monge–Ampère equation $\mathcal{E}$ is naturally equivalent to a pair of 2-dimensional, skew-othogonal non-lagrangian subdistributions $\mathcal{D}^1(\mathcal{E})$, $\mathcal{D}^2(\mathcal{E})$ of the Cartan distribution $\mathcal{C}_1$ in $J^1\tau$. In particular, the equivalence problem for hyperbolic Monge–Ampère equations with respect to contact transformations is the equivalence problem for pairs of 2-dimensional, skew-orthogonal non-lagrangian subdistributions of $\mathcal{C}_1$ with respect to contact transformations.

3.3. **Bundles of Monge–Ampère equations.** Beginning with this section, we set $M = J^1\tau$.

3.3.1. *Bundles of hyperbolic Monge–Ampère equations.* Let $\mathcal{E}$ be a Monge–Ampère equation (5). It is identified with the section

$$S_{\mathcal{E}} : x \mapsto \big[N(x) : A(x) : B(x) : C(x) : D(x)\big]$$

of the trivial bundle

$$\rho : \mathbb{R}\mathrm{P}^4 \times M \longrightarrow M\,, \quad \big([p^0 : p^1 : p^2 : p^3 : p^4],\, x\big) \mapsto x\,,$$

where $\mathbb{R}\mathrm{P}^4$ is the 4-dimensional projective space. Obviously, this identification is a bijection of the set of all Monge–Ampère equations onto the set of all sections of ρ.

Consider the open subset E of the total space of ρ defined by the condition (14), i.e.,

$$(p^2)^2 - 4p^1p^3 + 4p^4p^0 > 0\,.$$

Clearly, the section $S_{\mathcal{E}}$ corresponding to a hyperbolic Monge–Ampère equation $\mathcal{E}$ takes values in E. Thus we can define the bundle of hyperbolic Monge–Ampère equations by the formula

$$\pi = \rho\big|_E : E \longrightarrow M\,, \quad \big([p^0 : p^1 : p^2 : p^3 : p^4],\, x\big) \mapsto x\,.$$

We use local coordinates $x^1, \ldots, x^5, u^1, \ldots, u^4$ in the total space E of π, where $x^1 = x$, $x^2 = y$, $x^3 = z$, $x^4 = p$, $x^5 = q$ are the standard coordinates on M, while the coordinates $u^1, \ldots, u^4$ on the fibres of π are defined as follows. Consider the affine hyperplane in $\mathbb{R}^5$ defined by the equation $p^0 = 1$. It generates the local chart in E

$$[1 : p^1 : p^2 : p^3 : p^4] \mapsto (p^1, p^2, p^3, p^4)\,.$$

Following formulas (18), we introduce the local coordinates $u^1, \ldots, u^4$ along the fibres of π by

$$u^1 = -p^3\,, \quad u^2 = \frac{p^2 - \sqrt{\Delta}}{2}\,, \quad u^3 = \frac{p^2 + \sqrt{\Delta}}{2}\,, \quad u^4 = -p^1\,, \tag{19}$$

where $\Delta = (p^2)^2 - 4p^1p^3 + 4p^4$.

These coordinates induce the standard coordinates $x^j, u^i, u^i_{j_1}, \ldots, u^i_{j_1 \ldots j_k}$, where $1 \le j_1 \le \ldots \le j_r \le 5$, in the jet bundle $J^k\pi$.

3.3.2. *The lifting of contact transformations.* Let φ be a contact transformation defined in M. Then φ transforms any Monge–Ampère equation $\mathcal{E}$ to another Monge–Ampère equation $\tilde{\mathcal{E}}$. In other words, φ induces a transformation of the corresponding sections $S_{\mathcal{E}} \mapsto S_{\tilde{\mathcal{E}}}$. Therefore, the contact transformation φ induces a diffeomorphism $\varphi^{(0)}$ of the total space of π such that the diagram

$$\begin{tikzcd} E \arrow[r, "\varphi^{(0)}"] \arrow[d, "\pi"'] & E \arrow[d, "\pi"] \\ M \arrow[r, "\varphi"'] & M \end{tikzcd}$$

is commutative (in the domain of $\varphi^{(0)}$). The diffeomorphism $\varphi^{(0)}$ is called the *lifting of* φ *to the bundle* π.

The diffeomorphism $\varphi^{(0)}$, in its turn, can be lifted to a diffeomorphism $\varphi^{(k)}$ of $J^k\pi$ by the formula

$$\varphi^{(k)}(\,[S]^k_x\,) = \big[\,\varphi^{(0)} \circ S \circ \varphi^{-1}\,\big]^k_{\varphi(x)}\,.$$

Obviously, for any $l > m$, the diagram

$$\begin{tikzcd} J^l\pi \arrow[r, "\varphi^{(l)}"] \arrow[d, "\pi_{l,m}"'] & J^l\pi \arrow[d, "\pi_{l,m}"] \\ J^m\pi \arrow[r, "\varphi^{(m)}"'] & J^m\pi \end{tikzcd}$$

is commutative (in the domains of $\varphi^{(l)}$). The diffeomorphism $\varphi^{(k)}$ is called the *lifting of* φ *to the jet bundle* $J^k\pi$.

3.3.3. *The lifting of contact vector fields.* Let X be a contact vector field in M and let φ_t be its flow. Then $\varphi_t^{(k)}$ defines a vector field $X^{(k)}$ in $J^k\pi$. This field is called the *lifting of* X *to* $J^k\pi$. Obviously

$$(\,\pi_{l,m}\,)_*\big(\,X^{(l)}\,\big) = X^{(m)}\,, \;\; \infty \ge l > m \ge -1\,,$$

where $X^{(-1)} = X$.

It is not hard to prove that the map

$$X \longmapsto X^{(k)}$$

is a homomorphism of the Lie algebra of all contact vector fields onto the Lie algebra generated by all vector fields of the form $X^{(k)}$.

To express $X^{(k)}$ in the coordinates x^j, u^i_σ, we recall that the operator D_j of total derivative with respect to x^j is given by the formula

$$D_j = \frac{\partial}{\partial x^j} + \sum_{|\sigma|\geq 0} \sum_{i=1}^{4} u^i_{\sigma j} \frac{\partial}{\partial u^i_\sigma}\,, \; j = 1, 2, \ldots, 5\,.$$

The operator of evolution differentiation corresponding to a generating function $\psi(X) = (\psi^1(X), \ldots, \psi^4(X))^t$ is defined by the formula

$$\ni_{\psi(X)} = \sum_{|\sigma|\geq 0} \sum_{i=1}^{4} D_\sigma\big(\,\psi^i(X)\,\big) \frac{\partial}{\partial u^i_\sigma}\,,$$

where $\sigma = \{j_1 \ldots j_r\}$, $D_\sigma = D_{j_1} \circ \ldots \circ D_{j_r}$ and $\psi(X)$ is defined in the following way: Let S be a section of π defined in the domain of X, $\theta_1 = [S]^1_x$, and $x = \pi_1(\theta_1)$; then

$$\psi(X)(\theta_1) = \frac{d}{dt}\big(\,\varphi^{(0)}_t \circ S \circ \varphi^{-1}_t\,\big)\Big|_{t=0}(x)\,.$$

Now, suppose that

$$X = \sum_{i=1}^{5} X^i \frac{\partial}{\partial x^i}\,.$$

Then the lifting $X^{(\infty)}$ is defined by the formula (see [5])

$$X^{(\infty)} = \sum_{j=1}^{5} X^j D_j + \ni_{\psi(X)}\,.$$

From this formula, it follows that

$$X^{(k)} = \sum_{j=1}^{5} X^j D^k_j + \ni^k_{\psi(X)}\,, \tag{20}$$

where

$$D^k_j = \frac{\partial}{\partial x^j} + \sum_{0\leq|\sigma|\leq k} \sum_{i=1}^{4} u^i_{\sigma j} \frac{\partial}{\partial u^i_\sigma}\,, \quad \ni^k_{\psi(X)} = \sum_{0\leq|\sigma|\leq k} \sum_{i=1}^{4} D_\sigma\big(\,\psi^i(X)\,\big) \frac{\partial}{\partial u^i_\sigma}\,.$$

Let f be the generating function of the contact vector field X (see formula (2)) and $\theta_1 = (\,x, y, z, p, q, u^i, u^i_x, u^i_y, u^i_z, u^i_p, u^i_q\,)$. Then the vector $\psi(X_f)(\theta_1)$

is $(\psi^1,\dots,\psi^4)$, where

$$
\begin{aligned}
\psi^1 \;=\;& -fu^1_z - f_x u^1_p - f_y u^1_q + (u^1_y + qu^1_z)f_q + (u^1_x + pu^1_z)f_p \\
& + (-pu^1_p - qu^1_q + u^1)f_z + f_{p^2}(u^1)^2 + f_{q^2}u^2u^3 \\
& + (u^2+u^3)f_{pq}u^1 + 2f_{zp}pu^1 + 2f_{xp}u^1 + (u^2+u^3)f_{zq}p \\
& + (u^2+u^3)f_{xq} + f_{z^2}p^2 + 2f_{xz}p + f_{x^2}, \\
\psi^2 \;=\;& -fu^2_z - f_x u^2_p - f_y u^2_q + (u^2_y + qu^2_z)f_q + (u^2_x + pu^2_z)f_p \\
& + (-pu^2_p - qu^2_q + u^2)f_z + f_{p^2}u^1u^2 + f_{q^2}u^2u^4 + f_{yp}u^1 \\
& + f_{yq}u^2 + f_{xp}u^2 + f_{xq}u^4 + (qu^1 + pu^2)f_{zp} \\
& + (u^1u^4 + (u^2)^2)f_{pq} + (qu^2 + pu^4)f_{zq} + f_{z^2}pq + f_{yz}p \\
& + f_{xz}q + f_{xy}, \\
\psi^3 \;=\;& -fu^3_z - f_x u^3_p - f_y u^3_q + (u^3_y + qu^3_z)f_q + (u^3_x + pu^3_z)f_p \\
& + (-pu^3_p - qu^3_q + u^3)f_z + f_{p^2}u^1u^3 + f_{q^2}u^3u^4 + f_{yp}u^1 \\
& + f_{xp}u^3 + f_{yq}u^3 + f_{xq}u^4 + (qu^1 + pu^3)f_{zp} \\
& + (u^1u^4 + (u^3)^2)f_{pq} + (qu^3 + pu^4)f_{zq} + f_{z^2}pq + f_{yz}p \\
& + f_{xz}q + f_{xy}, \\
\psi^4 \;=\;& -fu^4_z - f_x u^4_p - f_y u^4_q + (u^4_y + qu^4_z)f_q + (u^4_x + pu^4_z)f_p \\
& + (-pu^4_p - qu^4_q + u^4)f_z + f_{p^2}u^2u^3 + f_{q^2}(u^4)^2 \\
& + (u^2+u^3)f_{pq}u^4 + 2f_{yq}u^4 + 2f_{zq}qu^4 + (u^2+u^3)f_{yp} \\
& + (u^2+u^3)f_{zp}q + f_{z^2}q^2 + 2f_{yz}q + f_{y^2}.
\end{aligned}
\tag{21}
$$

3.4. **Differential invariants.** By Γ we denote the pseudogroup of all contact transformations of the base M of π. It acts on every $J^k\pi$ by its lifted diffeomorphisms.

A function or a vector field or a differential form or any other object defined in $J^k\pi$ is a *differential invariant of the action of* Γ *on* $J^k\pi$ if for any $\varphi \in \Gamma$ the lifted transformation $\varphi^{(k)}$ preserves this object. In this work, these differential invariants are called also *differential invariants (of order k) of Monge–Ampère equations* or simply *differential invariants (of order k)*.

Let $\mathcal{E}$ be a Monge–Ampère equation, $S_{\mathcal{E}}$ the section of π identified with $\mathcal{E}$, and I a differential invariant of order k. Then the restriction $I|_{L^{(k)}_{S_{\mathcal{E}}}}$ is denoted by $I_{\mathcal{E}}$. If a contact transformation f transforms $\mathcal{E}$ to $\tilde{\mathcal{E}}$, then obviously $f^{(k)}$ transforms $I_{\mathcal{E}}$ to $I_{\tilde{\mathcal{E}}}$, for any kth order invariant I.

Functions that are differential invariants are also called *scalar differential invariants.* By A_k we denote the set of all scalar differential invariants of order $\le k$. It is clear that A_k is an $\mathbb{R}$-algebra. Then we have a sequence of inclusions

$$A_0 \subset A_1 \subset \ldots \subset A_k \subset A_{k+1} \subset \ldots$$

The $\mathbb{R}$-algebra $A = \bigcup_{k=0}^{\infty} A_k$ is called the *algebra of scalar differential invariants of Monge–Ampère equations.*

Let X be a contact vector field in M and I a differential invariant of order k. Then $L_{X^{(k)}}(I) = 0$, where L stands for the Lie derivative. This means, in particular, that kth order scalar invariants are 1st integrals of all contact vector fields lifted to $J^k\pi$. This gives the well-known general method to calculate scalar differential invariants.

Obviously, a scalar differential invariant of order k is constant on every orbit of the action Γ on $J^k\pi$.

Proposition 3.3. *(1) $J^k\pi$ is an orbit of the action of Γ iff $k = 0, 1$,*
(2) Codimension of a generic orbit of $J^2\pi$ is equal to 2.
(3) Codimension of a generic orbit of $J^3\pi$ is equal to 29.

Proof. Let θ_k be a generic point of $J^k\pi$ and Orb_{θ_k} the orbit of the action Γ on $J^k\pi$ passing through θ_k. Then $\operatorname{codim} \mathrm{Orb}_{\theta_k} = \dim J^k\pi - \dim \mathrm{Orb}_{\theta_k}$. The dimension of Orb_{θ_k} is the dimension of the subspace spanned by all vectors $X^{(k)}(\theta_k)$. It can be calculated with the help of computer algebra using formulas (20) and (21). □

From this proposition, we obtain immediately

Corollary 3.4. *(1) The algebra of scalar differential invariants A_2 is generated by 2 functionally independent invarints.*
(2) The algebra of scalar differential invariants A_3 is generated by 29 functionally independent invarints.

4. Differential invariants on $J^2\pi$

4.1. **Base projectors.** Since we consider a generic hyperbolic Monge–Ampère equation $\mathcal{E}$, then

$$\dim(\mathcal{D}^1(\mathcal{E}))^{(1)} = \dim(\mathcal{D}^2(\mathcal{E}))^{(1)} = 3$$

and the distributions $(\mathcal{D}^1(\mathcal{E}))^{(1)}$, $(\mathcal{D}^2(\mathcal{E}))^{(1)}$ are generic. It follows that

$$\dim(\mathcal{D}^1(\mathcal{E}))^{(2)} = \dim(\mathcal{D}^2(\mathcal{E}))^{(2)} = 5\,.$$

Suppose that vector fields X_1, X_2 generate the distribution $\mathcal{D}^1(\mathcal{E})$ and vector fields X_3, X_4 generate the distribution $\mathcal{D}^2(\mathcal{E})$. The 3-dimensional generic distributions $\langle X_1, X_2, [X_1, X_2]\rangle$ and $\langle X_3, X_4, [X_3, X_4]\rangle$ intersect along a one-dimensional subdistribution $\mathcal{D}^3(\mathcal{E}) = \langle X_1, X_2, [X_1, X_2]\rangle \cap \langle X_3, X_4, [X_3, X_4]\rangle$. Hence, equation $\mathcal{E}$ generates a direct sum decomposition

$$T(M) = \mathcal{D}^1(\mathcal{E}) \oplus \mathcal{D}^2(\mathcal{E}) \oplus \mathcal{D}^3(\mathcal{E}). \tag{22}$$

This decomposition generates three projections

$$\mathcal{P}_i : T(M) \to \mathcal{D}^i(\mathcal{E})\,, \quad i = 1, 2, 3\,.$$

These projections can be viewed as vector-valued 1-forms in the following way. Suppose X_5 is a vector field generating $\mathcal{D}^3$ and a coframe field $\{\omega^1, \dots, \omega^5\}$ on M is generated by a frame field $\{X_1, \dots, X_5\}$ on M. Then

$$\begin{aligned} \mathcal{P}_1 &= \omega^1 \otimes X_1 + \omega^2 \otimes X_2\,, \\ \mathcal{P}_2 &= \omega^3 \otimes X_3 + \omega^4 \otimes X_4\,, \\ \mathcal{P}_3 &= \omega^5 \otimes X_5\,. \end{aligned} \tag{23}$$

It is clear that these vector-valued differential 1-forms are differential invariants of $\mathcal{E}$ with respect to contact transformations.

Obviously, the initial equation $\mathcal{E}$ can be reconstructed completely from each of the invariants $\mathcal{P}_1$ and $\mathcal{P}_2$.

4.2. **Coordinate expressions for base projectors.** To calculate in coordinates, we fix the vector fields $X_1, \dots, X_5$ according to equations (18) and (19), that is,

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x} + p\frac{\partial}{\partial z} + u^1\frac{\partial}{\partial p} + u^2\frac{\partial}{\partial q}\,, \ X_2 = \frac{\partial}{\partial y} + q\frac{\partial}{\partial z} + u^3\frac{\partial}{\partial p} + u^4\frac{\partial}{\partial q}\,, \\ X_3 &= \frac{\partial}{\partial x} + p\frac{\partial}{\partial z} + u^1\frac{\partial}{\partial p} + u^3\frac{\partial}{\partial q}\,, \ X_4 = \frac{\partial}{\partial y} + q\frac{\partial}{\partial z} + u^2\frac{\partial}{\partial p} + u^4\frac{\partial}{\partial q}\,. \end{aligned} \tag{24}$$

Obviously,

$$X_5 = \lambda^1 X_1 + \lambda^2 X_2 + \kappa[X_1, X_2] = \lambda^3 X_3 + \lambda^4 X_4 + \chi[X_3, X_4]\,. \tag{25}$$

By easy calculation we get

$$\lambda^3 = \lambda^1\,, \quad \lambda^4 = \lambda^2\,, \quad \chi = -\kappa \neq 0\,,$$

where

$$\begin{aligned} \lambda^1 = \frac{1}{u^2 - u^3}\Big(&(u^2 + u^3)_y + q(u^2 + u^3)_z + u^4(u^2 + u^3)_q \\ &- 2(u^4_x + pu^4_z + u_1 u^4_p) - (u^2 + u^3)u^4_q + u^3u^2_p + u^2u^3_p\Big)\,, \end{aligned} \tag{26}$$

$$\begin{aligned} \lambda^2 = \frac{1}{u^2 - u^3}\Big(&(u^2 + u^3)_x + p(u^2 + u^3)_z + u^1(u^2 + u^3)_p \\ &- 2(u^1_y + qu^1_z + u_4 u^1_q) - (u^2 + u^3)u^1_p + u^2u^3_q + u^3u^2_q\Big) \end{aligned} \tag{27}$$

provided we normalize X_5 by the requirement $\kappa = 1$.

Consider the brackets of the vector fields $X_1, \dots, X_5$. We have

$$[X_j, X_k] = \sum_{i=1}^{5} b^i_{jk} X_i\,.$$

Obviously,

$$b^i_{jk} = -b^i_{kj}\,. \tag{28}$$

From (25) and $\kappa = 1$, we have

$$\begin{aligned} &b^1_{12} = -\lambda^1\,, \quad b^2_{12} = -\lambda^2\,, \quad b^3_{12} = 0\,, \quad b^4_{12} = 0\,, \quad b^5_{12} = 1\,, \\ &b^1_{34} = 0\,, \quad b^2_{34} = 0\,, \quad b^3_{34} = \lambda^1\,, \quad b^4_{34} = \lambda^2\,, \quad b^5_{34} = -1\,. \end{aligned} \tag{29}$$

By our definition, the form ω^5 is a contact form on M. Then $d\omega^5$ is a symplectic form and $\langle X_1, X_2\rangle$ and $\langle X_3, X_4\rangle$ are skew orthogonal with respect to this form. It follows

$$b^5_{13} = b^5_{14} = 0\,, \quad b^5_{23} = b^5_{24} = 0\,. \tag{30}$$

Consider the differentials of functions b^i_{jk}. We define $c^i_{jk,r}$ as coefficients in the decomposition

$$db^i_{jk} = \sum_{r=1}^{5} c^i_{jk,r}\omega^r\,.$$

From (28), we have

$$c^i_{jk,r} = -c^i_{kj,r}\,. \tag{31}$$

From (29) and (30), we get

$$\begin{aligned} & c^1_{12,r} = -\lambda^1_r\,, && c^2_{12,r} = -\lambda^2_r\,, && c^3_{12,r} = 0\,, && c^4_{12,r} = 0\,, && c^5_{12,r} = 0\,, \\ & c^1_{34,r} = 0\,, && c^2_{34,r} = 0\,, && c^3_{34,r} = \lambda^1_r\,, && c^4_{34,r} = \lambda^2_r\,, && c^5_{34,r} = 0\,, \\ & c^5_{13,r} = 0\,, && c^5_{14,r} = 0\,, && c^5_{23,r} = 0\,, && c^5_{24,r} = 0\,, \end{aligned} \tag{32}$$

where the functions λ^1_r and λ^2_r are defined by

$$d\lambda^1 = \sum_{r=1}^{5} \lambda^1_r \omega^r\,, \quad d\lambda^2 = \sum_{r=1}^{5} \lambda^2_r \omega^r\,.$$

4.3. **Curvatures.** Using formulas (3), (4) and the direct sum decomposition (22), it is easy to compute the curvature forms of the distributions $\mathcal{D}^1_{\mathcal{E}}$, $\mathcal{D}^2_{\mathcal{E}}$, $\mathcal{D}^1_{\mathcal{E}} \oplus \mathcal{D}^3_{\mathcal{E}}$, $\mathcal{D}^1_{\mathcal{E}} \oplus \mathcal{D}^3_{\mathcal{E}}$, and $\mathcal{C}$, which are

$$\begin{aligned} \mathcal{R}_1 &= -\omega^1 \wedge \omega^2 \otimes X_5\,, \\ \mathcal{R}_2 &= \omega^3 \wedge \omega^4 \otimes X_5\,, \\ \mathcal{R}^1_1 &= -(b^3_{15}\omega^1 + b^3_{25}\omega^2) \wedge \omega^5 \otimes X_3 - (b^4_{15}\omega^1 + b^4_{25}\omega^2) \wedge \omega^5 \otimes X_4\,, \\ \mathcal{R}^1_2 &= -(b^1_{35}\omega^3 + b^1_{45}\omega^4) \wedge \omega^5 \otimes X_1 - (b^2_{35}\omega^3 + b^2_{45}\omega^4) \wedge \omega^5 \otimes X_2\,, \\ \mathcal{R} &= \mathcal{R}_1 + \mathcal{R}_2\,, \end{aligned} \tag{33}$$

respectively. It is clear that these curvature forms are differential invariants of $\mathcal{E}$ with respect to contact transformations.

Proposition 4.1. *Every* 2*-form* $[\![\mathcal{P}_i, \mathcal{P}_j]\!]$, $i, j = 1, 2, 3$, *is a linear combination of the curvature forms.*

Proof. By direct calculation we obtain formulas

$$\begin{aligned} [\![\mathcal{P}_1, \mathcal{P}_2]\!] &= \tfrac{1}{2}(-[\![\mathcal{P}_1, \mathcal{P}_1]\!] - [\![\mathcal{P}_2, \mathcal{P}_2]\!] + [\![\mathcal{P}_3, \mathcal{P}_3]\!])\,, \\ [\![\mathcal{P}_1, \mathcal{P}_3]\!] &= \tfrac{1}{2}(-[\![\mathcal{P}_1, \mathcal{P}_1]\!] + [\![\mathcal{P}_2, \mathcal{P}_2]\!] - [\![\mathcal{P}_3, \mathcal{P}_3]\!])\,, \\ [\![\mathcal{P}_2, \mathcal{P}_3]\!] &= \tfrac{1}{2}([\![\mathcal{P}_1, \mathcal{P}_1]\!] - [\![\mathcal{P}_2, \mathcal{P}_2]\!] - [\![\mathcal{P}_3, \mathcal{P}_3]\!]) \end{aligned}$$

and

$$[\![\mathcal{P}_1, \mathcal{P}_1]\!] = -2(\mathcal{R}^1_2 + \mathcal{R}_1), \quad [\![\mathcal{P}_2, \mathcal{P}_2]\!] = -2(\mathcal{R}^1_1 + \mathcal{R}_2), \quad [\![\mathcal{P}_3, \mathcal{P}_3]\!] = -2(\mathcal{R}_1 + \mathcal{R}_2).$$

This completes the proof. □

4.4. **Scalar invariants on $J^2\pi$.** We consider the following three invariant 5-forms with values in $\mathcal{D}^3_{\mathcal{E}} = \langle X_5 \rangle$:

$$\begin{aligned} \tfrac{1}{2}\big(\mathcal{R}^1_2 \lrcorner \mathcal{R}_1\big) \lrcorner \big(\mathcal{R}^1_2 \lrcorner \mathcal{R}_1\big) &= (b^2_{35}b^1_{45} - b^1_{35}b^2_{45})\,\omega^1 \wedge \ldots \wedge \omega^5 \otimes X_5\,, \\ \tfrac{1}{2}\big(\mathcal{R}^1_1 \lrcorner \mathcal{R}_2\big) \lrcorner \big(\mathcal{R}^1_1 \lrcorner \mathcal{R}_2\big) &= (b^4_{15}b^3_{25} - b^3_{15}b^4_{25})\,\omega^1 \wedge \ldots \wedge \omega^5 \otimes X_5\,, \\ \big(\mathcal{R}^1_2 \lrcorner \mathcal{R}_1\big) \lrcorner \big(\mathcal{R}^1_1 \lrcorner \mathcal{R}_2\big) &= (b^3_{15}b^1_{35} + b^4_{15}b^1_{45} \\ &\quad + b^3_{25}b^2_{35} + b^4_{25}b^2_{45})\,\omega^1 \wedge \ldots \wedge \omega^5 \otimes X_5\,. \end{aligned} \tag{34}$$

Obviously, coefficients of proportionality between them are scalar differential invariants. Therefore, assuming

$$b^3_{15}b^1_{35} + b^4_{15}b^1_{45} + b^3_{25}b^2_{35} + b^4_{25}b^2_{45} \neq 0\,, \tag{35}$$

we get the following scalar differential invariants:

$$
\begin{aligned}
I^1 &= \frac{b^2_{35}b^1_{45} - b^1_{35}b^2_{45}}{b^3_{15}b^1_{35} + b^4_{15}b^1_{45} + b^3_{25}b^2_{35} + b^4_{25}b^2_{45}}\,,\\
I^2 &= \frac{b^4_{15}b^3_{25} - b^3_{15}b^4_{25}}{b^3_{15}b^1_{35} + b^4_{15}b^1_{45} + b^3_{25}b^2_{35} + b^4_{25}b^2_{45}}\,.
\end{aligned} \tag{36}
$$

are scalar differential invariants on $J^2\pi$.

Proposition 4.2. *The scalar invariants I_1 and I_2 are functionally independent.*

Proof. By coordinate calculation. □

Using Corollary 3.4, we get

Theorem 4.3. *The algebra of scalar differential invariants on $J^2\pi$ is generated by the invariants I^1 and I^2.*

5. Differential invariants on $J^3\pi$

5.1. **The complete parallelism.** Applying the exterior differential to scalar invariants I^1 and I^2, we obtain the invariant differential 1-forms dI^1 and dI^2 on $J^3\pi$. It makes possible to construct the following invariant differential 1-forms on $J^3\pi$:

$$
\begin{aligned}
\Omega^1 &= \mathcal{P}_1 \rfloor dI^1 = X_1(I^1)\omega^1 + X_2(I^1)\omega^2\,,\\
\Omega^2 &= \mathcal{P}_1 \rfloor dI^2 = X_1(I^2)\omega^1 + X_2(I^2)\omega^2\,,\\
\Omega^3 &= \mathcal{P}_2 \rfloor dI^1 = X_3(I^1)\omega^3 + X_4(I^1)\omega^4\,,\\
\Omega^4 &= \mathcal{P}_2 \rfloor dI^2 = X_3(I^2)\omega^3 + X_4(I^2)\omega^4\,,\\
\Omega^5_1 &= \mathcal{P}_3 \rfloor dI^1 = X_5(I^1)\omega^5\,, \quad \Omega^5_2 = \mathcal{P}_1 \rfloor dI^2 = X_5(I^2)\omega^5\,.
\end{aligned} \tag{37}
$$

We suppose that $\mathcal{E}$ is a generic equation. In particular, we may assume that

$$
X_5(I^1) \neq 0\,, \quad X_5(I^2) \neq 0\,, \tag{38}
$$

and

$$
\Delta_1 = \begin{vmatrix} X_1(I^1) & X_2(I^1) \\ X_1(I^2) & X_2(I^2) \end{vmatrix} \neq 0\,, \quad \Delta_2 = \begin{vmatrix} X_3(I^1) & X_4(I^1) \\ X_3(I^2) & X_4(I^2) \end{vmatrix} \neq 0\,. \tag{39}
$$

This means that the collections of differential 1-forms $\{\Omega^1, \ldots, \Omega^4, \Omega^5_1\}$ and $\{\Omega^1, \ldots, \Omega^4, \Omega^5_2\}$ are invariant coframes in M. Each of these coframes defines the invariant complete parallelism on M.

The following invariant frames $\{Y_1,\dots,Y_4,Y_5^1\}$ and $\{Y_1,\dots,Y_4,Y_5^2\}$, where

$$\begin{aligned}
Y_1 &= \frac{1}{\Delta_1}\big(X_2(I^2)X_1 - X_1(I^2)X_2\big)\,,\\
Y_2 &= \frac{1}{\Delta_1}\big(-X_2(I^1)X_1 + X_1(I^1)X_2\big)\,,\\
Y_3 &= \frac{1}{\Delta_2}\big(X_4(I^2)X_3 - X_3(I^2)X_4\big)\,,\\
Y_4 &= \frac{1}{\Delta_2}\big(-X_4(I^1)X_3 + X_3(I^1)X_4\big)\,,\\
Y_5^1 &= \frac{1}{X_5(I^1)}X_5\,,\quad Y_5^2 = \frac{1}{X_5(I^2)}X_5\,,
\end{aligned} \tag{40}$$

are dual to the above-mentioned coframes.

5.2. **Scalar invariants on $J^3\pi$.**

5.2.1. The invariant 1-forms Ω_1^5 and Ω_2^5 obtained above are contact forms on M. Therefore the coefficient of proportionality

$$I^3 = \frac{X_5(I^1)}{X_5(I^2)}$$

between these forms is a scalar differential invariant on $J^3\pi$.

We already obtained numerous invariant objects of third order: functions, differential form, differential vector valued form, and vector fields. Applying operations of tensor algebra and Frölicher–Nijenhuis brackets to these objects, we can construct new invariant objects of third order the same as above.

Many scalar differential invariants can be obtained as coefficients of linear dependence between these objects. In this way, we already obtained the invariants I^1, I^2, I^3.

Another way to construct scalar invariants uses expression of invariant objects with respect to an invariant base. Components of an invariant object with respect to an invariant base are scalar differential invariants.

Finally, invariant objects can possess their own scalar invariants. For example, the determinant, the trace, and the other coefficients of the characteristic polynomial of an invariant linear operator are scalar differential invariants. Consider the invariant linear operators

$$\Delta_1 = Y_5 \lrcorner \mathcal{R}_1^1 : D \to D_{\mathcal{D}_\mathcal{E}^2}\,,\quad \Delta_2 = Y_5 \lrcorner \mathcal{R}_2^1 : D \to D_{\mathcal{D}_\mathcal{E}^1}\,,$$

where D is the module of all vector fields on M and $D_{\mathcal{D}_\mathcal{E}^1}$ and $D_{\mathcal{D}_\mathcal{E}^2}$ are its submodules of all vector fields belonging to $\mathcal{D}_\mathcal{E}^1$ and $\mathcal{D}_\mathcal{E}^2$ respectively. Consider the invariant operator

$$\nabla_1 = \Delta_2\big|_{D_{\mathcal{D}_\mathcal{E}^2}} \circ \Delta_1\big|_{D_{\mathcal{D}_\mathcal{E}^1}} : D_{\mathcal{D}_\mathcal{E}^1} \to D_{\mathcal{D}_\mathcal{E}^1}\,.$$

Using (33) and (40), we get the following scalar differential invariants :

$$\begin{aligned} I^4 &= \operatorname{tr}(\nabla_1) = \frac{1}{X_5(I^1)^2}(b^3_{15}b^1_{35} + b^4_{15}b^1_{45} + b^3_{25}b^2_{35} + b^4_{25}b^2_{45})\,, \\ I^5 &= \det(\nabla_1) = \frac{1}{X_5(I^1)^4}(b^2_{35}b^1_{45} - b^1_{35}b^2_{45})(b^4_{15}b^3_{25} - b^3_{15}b^4_{25})\,. \end{aligned} \tag{41}$$

5.2.2. Consider the invariant operator

$$\nabla_2 = \Delta_1\big|_{D_{\mathcal{D}^1_{\mathcal{E}}}} \circ \Delta_2\big|_{D_{\mathcal{D}^2_{\mathcal{E}}}} : D_{\mathcal{D}^2_{\mathcal{E}}} \to D_{\mathcal{D}^2_{\mathcal{E}}}\,.$$

The characteristic polynomials of this operator and the operator ∇_1 are the same.

Suppose v_1 and v_2 are eigenvectors for the operators ∇_1 and ∇_2 respectively. Then obviously, $\Delta_1(v_1)$ and $\Delta_2(v_2)$ are eigenvectors for ∇_2 and ∇_1 respectively.

From (36) and (41), we get that the discriminant of the characteristic polynomial of the operator ∇_i is

$$\frac{(b^3_{15}b^1_{35} + b^4_{15}b^1_{45} + b^3_{25}b^2_{35} + b^4_{25}b^2_{45})^2}{X_5(I^1)^4}\,(1 - 4I^1I^2)\,.$$

Thus from (35) and (38), we obtain

Theorem 5.1. (1) *The operator ∇_i has two different eigenfunctions, if* $1 - 4I^1I^2 > 0$.
(2) *The operator ∇_i has a unique eigenfunction, if* $1 - 4I^1I^2 = 0$.
(3) *The operator ∇_i has no eigenfunctions, if* $1 - 4I^1I^2 < 0$.

Thus, the set of all generic hyperbolic Monge–Ampère equations is divided onto three types: $\mathcal{E}$ has type "h" if $1-4I^1_{\mathcal{E}}I^2_{\mathcal{E}} > 0$, $\mathcal{E}$ has type "p" if $1-4I^1_{\mathcal{E}}I^2_{\mathcal{E}} = 0$, and $\mathcal{E}$ has type "e" if $1 - 4I^1_{\mathcal{E}}I^2_{\mathcal{E}} < 0$.

5.2.3. Return to a construction of scalar differential invariants. Let us use the above-mention second way to construct these invariants. Consider invariant vector valued 2-forms on $J^3\pi$. From proposition 4.1 they are linear combinations of the curvature forms. Therefore we have 8 invariant 2-forms

$$\begin{aligned} \mathcal{R}_1 \lrcorner\, dI^1 &= I^6\Omega^1 \wedge \Omega^2\,, \\ \mathcal{R}_1 \lrcorner\, dI^2 &= I^7\Omega^1 \wedge \Omega^2\,, \\ \mathcal{R}_2 \lrcorner\, dI^1 &= I^8\Omega^3 \wedge \Omega^4\,, \\ \mathcal{R}_2 \lrcorner\, dI^2 &= I^9\Omega^3 \wedge \Omega^4\,, \\ \mathcal{R}^1_1 \lrcorner\, dI^1 &= I^{10}\Omega^1 \wedge \Omega^5 + I^{11}\Omega^2 \wedge \Omega^5\,, \\ \mathcal{R}^1_1 \lrcorner\, dI^2 &= I12\Omega^1 \wedge \Omega^5 + I^{13}\Omega^2 \wedge \Omega^5\,, \\ \mathcal{R}^1_2 \lrcorner\, dI^1 &= I^{14}\Omega^3 \wedge \Omega^5 + I^{15}\Omega^4 \wedge \Omega^5\,, \\ \mathcal{R}^1_2 \lrcorner\, dI^2 &= I^{16}\Omega^3 \wedge \Omega^5 + I^{17}\Omega^4 \wedge \Omega^5\,, \end{aligned}$$

Taking into account the invariant I^3, from the first two equations, we obtain the following scalar differential invariants on $J^3\pi$

$$I^6 = \frac{\Delta_1}{X_5(I^1)}\,, \quad I^7 = \frac{\Delta_2}{X_5(I^1)}\,.$$

By the same way we can construct new scalar differential invariants using invariant vector valued 3-forms, 4-forms, and 5-forms. For example the 3-forms $[\![\mathcal{P}_i, \mathcal{R}_j]\!]$ or $[\![\mathcal{P}_i, \mathcal{R}^1_j]\!]$, the 4-forms $[\![\mathcal{P}_i, (\mathcal{R}^1_j \lrcorner \mathcal{R}^1_k)]\!]$, the 5-forms $[\![\mathcal{P}_i, \mathcal{R}^1_j]\!] \lrcorner [\![\mathcal{P}_k, \mathcal{R}^1_l]\!]$, etc. According to "the principle of n-invariants", it is enough to know five functionally independent scalar invariants $\mathcal{I}^1, \dots, \mathcal{I}^5$ on $J^\infty\pi$ to solve the equivalence problem for Monge–Ampère equations.

The following statement can be proved by calculations in coordinates.

Theorem 5.2. *Invariants I^1, I^2, I^3, I^4, and I^5 are functionally independent.*

6. The equivalence problem

Let $\mathcal{E}$ be generic hyperbolic Monge–Ampère equation considering as a section of the bundle π. Then the invariants $I^1_{\mathcal{E}}, \dots, I^5_{\mathcal{E}}$ form the coordinate system in M. In terms of these coordinates, the 1-forms $\Omega_1, \dots, \Omega_5$ defining the complete parallelism on M have the form

$$\Omega_i = \sum_{j=1}^{5} \Omega^i_j(I^1_{\mathcal{E}}, \dots, I^5_{\mathcal{E}}) dI^j\,, \quad i = 1, \dots, 5\,.$$

Theorem 6.1. *The equivalence class of the equation $\mathcal{E}$ with respect to contact transformations is defined by the functions $\Omega^i_j(I^1_{\mathcal{E}}, \dots, I^5_{\mathcal{E}})$ uniquely.*

Proof. Let $\tilde{\mathcal{E}}$ be another Monge–Ampère equation such that there exist a contact transformation f transforming it to $\mathcal{E}$. Then obviously that the functions $\Omega^i_j(I^1_{\mathcal{E}}, \dots, I^5_{\mathcal{E}})$ and $\tilde{\Omega}^i_j(I^1_{\tilde{\mathcal{E}}}, \dots, I^5_{\tilde{\mathcal{E}}})$ are the same for all i and j.

Let $\tilde{\mathcal{E}}$ be a Monge–Ampère equation such that the functions $\Omega^i_j(I^1_{\mathcal{E}}, \dots, I^5_{\mathcal{E}})$ and $\tilde{\Omega}^i_j(I^1_{\tilde{\mathcal{E}}}, \dots, I^5_{\tilde{\mathcal{E}}})$ are the same for all i and j and let. Suppose $x = (x^1, \dots, x^5)$ is the the standard coordinate system in M, $I_{\mathcal{E}} = (I^1_{\mathcal{E}}, \dots, I^5_{\mathcal{E}})$ and $I_{\tilde{\mathcal{E}}} = (I^1_{\tilde{\mathcal{E}}}, \dots, I^5_{\tilde{\mathcal{E}}})$ are invariant coordinate systems in M for $\mathcal{E}$ and $\tilde{\mathcal{E}}$ respectively. Then the transformation $x \circ I^{-1}_{\tilde{\mathcal{E}}} \circ I_{\mathcal{E}} \circ x$ is contact and it transforms $\tilde{\mathcal{E}}$ to $\mathcal{E}$. □

7. Coordinates in the bundle of Monge–Ampère equations

In this section, we introduce local coordinates that are convenient to calculate the above obtained differential invariants.

7.1. Several proofs in this paper depend on calculations, which are simply impossible to do in terms of standard jet coordinates on $J^k(\pi)$. This is why we suggest a different set of coordinates on $J^k(\pi)$.

Lemma 7.1. *Denote*

$$u^k_{i_1 \dots i_h} = \mathcal{X}_{i_1} \dots \mathcal{X}_{i_h} u^k, \quad i_1 \le \dots \le i_h. \tag{42}$$

Functions $x^i, u^k, u^k_{i_1\ldots i_h}$*,* $i_1 \le \ldots \le i_h$*,* $h \le n$*, constitute a coordinate system on* $J^n(\pi)$*. Moreover, the standard jet coordinates on* $J^n(\pi)$ *are rational functions of the coordinates* (42)*.*

Proof. The statement is directly verifiable for $n = 2$. To express $D_{i_1\ldots i_n}u^k$ in terms of coordinates (42) for $n > 2$, one exploits the following obvious facts: Firstly, fields D_i are linear combinations of fields $\mathcal{X}_i$ with coefficients from $C^\infty J^1(\pi)$; secondly, the coefficients $\mathcal{B}^j_{i_1 i_2}$ are functions on $J^2(\pi)$; thirdly $\mathcal{X}_{i_2}\mathcal{X}_{i_1}f = -\mathcal{B}^j_{i_1 i_2}\mathcal{X}_j f + \mathcal{X}_{i_1}\mathcal{X}_{i_2}f$ for every function a from $C^\infty J^n(\pi)$. □

Every invariant obtained so far was expressible in terms of the quantities b^k_{ij} and $c^k_{ij,r}$.

Below, we construct coordinate system in J^3 generated by functionally independent coefficients b^k_{ij} and $c^k_{ij,r}$.....

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